



Satisfaction-Oriented Fuzzy Reasoning

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Abstract. This paper examines the traditional architecture of fuzzy expert reasoning and suggests a reorientation, which is simpler, more intuitive, and more implementable. This new approach centers on first determining the degree of satisfaction that a rule applies to the current situation, and then using this degree of satisfaction to qualify the rule's conclusion. This provides a theoretical basis for the effective work being done in practical fuzzy expert systems. But beyond this, satisfaction-oriented fuzzy reasoning establishes a framework for a large new space of fuzzy reasoning, both theoretical and practical.



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MEYER'S LAW: [1]

It is a simple task to make things complex, but a complex task to make them simple.

1. INTRODUCTION

Reorienting one's view provides not only perspective and depth, but often reveals structural insight previously obscured. Despite the advantages of fuzzy reasoning, its propagation has been impeded, possibly by the lack of agreement on the specific formulation. This paper suggests a reorientation of the framework of fuzzy reasoning, which opens up a large new world of fuzzy reasoning beyond current systems for exploration both theoretical and practical.

The field of fuzzy logic and fuzzy expert systems has excited researchers for over 25 years. This interest has manifested both in a collection of research papers and experimental implementations. Zadeh suggested a framework for constructing fuzzy inferences called "the compositional rule of inference" [18], and this approach has been followed by most of the theoretical community. In examining both the theoretical work and the practical implementations, we found that a fuzzy implementation constructed by Togai InfraLogic, Inc. [8], [14] did not fit within the theoretical framework described in the papers. In analyzing this discrepancy, it was discovered that the theoretical system fundamentally differed from the reasoning of crisp expert systems, whereas the Togai implementation, and virtually all other implementations, aligned. However, Togai's system seemed ad hoc and the result of experimental intuition. In this paper, we construct a new theoretical framework, place Togai's work within it, and compare this new system with the classical approach, showing how the traditional theory may have misapplied ideas of logic into the realm of expert systems. In an example to be presented, both approaches are demonstrated; the new approach is seen to be simpler, more intuitive, and more implementable.

Fuzzy reasoning has many advantages over normal (“crisp”) reasoning; a crisp rule makes a curt decision based on a simple boolean test, as

if (satellite_temperature > 80°) **then** set_cooling_system_to 20°

In contrast, a fuzzy rule makes a more gradual and shaded judgement, as

if (satellite_temperature is HOT) **then** cooling_system_setting is HIGH

This judgement about the current temperature being HOT is not a simple test that yields “yes” or “no”, but rather an operation that yields a *degree* of truthfulness, represented as a real number between 0.0 and 1.0, inclusive. 0.0 represent complete falsity, and 1.0 represents complete truth; values in between represent an intermediate degree of satisfaction. HOT would be defined as a function which would take the current satellite temperature and return a fuzzy value between 0.0 and 1.0 denoting the degree to which the temperature was hot.

Likewise, the conclusion about setting the cooling system on HIGH does not specify a specific temperature to set the thermostat. Rather, HIGH represents a range of settings that are associated with a high degree of cooling. It would be defined as a function that takes possible values that the cooling system could be set to, and return a fuzzy value denoting the degree to which that setting is a high setting.

This ability to use descriptive nomenclature for qualities, such as HOT, without needing to pin the definition down to a rigid interval, both matches the intuition of the experts whose knowledge is being pooled to form the expert system, and also yields a more natural, gradual approach to reasoning, with more graceful adaptation to unforeseen situations. Crisp rule systems can only simulate this behavior using many crisp rules in place of each fuzzy rule, and even then the behavior is stilted and jerky.

In this paper, we will discuss both the classical and satisfaction approaches to performing fuzzy inferencing. To focus on inferencing, we do not discuss fuzzification or defuzzification, or chains of inference. But before we describe the structure of the two approaches, we need to first define the notation that we will use, and the useful concept of triangular norms.

2. DEFINITIONS AND NOTATION

The field of fuzzy logic originated from the work of L. A. Zadeh [17], as an extension of classical logic. It has also been used to aid in the management of uncertainty in expert systems [19]. A survey of some of the articles in the field reveals an enormous diversity of approaches and techniques, particularly for expert systems. The following description introduces some new notation, including the use of lambda notation to describe fuzzy sets.

We use the notation $D \rightarrow R$ to describe the *function type* from domain D to range R. A function f of type $D \rightarrow R$ may be described using *lambda notation* as

$$f = \lambda d. (\text{expression involving } d)$$

where d is a variable of type D, and the expression has type R. The λ symbol is just notation to denote a function, with its formal parameter list and its body. This means the same thing as

$$f(d) = (\text{expression involving } d)$$

For example, $\lambda x. x+3$ is the function that adds three to its argument; $f(x) = x+3$ defines the same function as $f = \lambda x. x+3$. This function $\lambda x. x+3$ is applied to 5 as $f(5) = (\lambda x. x+3)(5) = (\text{let } x=5 \text{ in } x+3) = 5+3 = 8$. As an axiom, $f = \lambda d. f(d)$ and $(\lambda d. f(d))(d_0) = f(d_0)$.

We will write the maximum value of an expression over different values of a variable as

$$\bigcup_{u \in U} (\text{expression involving } u)$$

A *fuzzy set* A is defined over a *universe of discourse* U , which is itself a normal set. A fuzzy set differs from a normal set in that all the members of U are considered to be in the fuzzy set, but each only to a particular degree, between 0 and 1. The fuzzy set A is described by a characteristic *fuzzy membership function*, which has domain U and range $[0,1]$. We identify the fuzzy set A with its membership function, so that we will simply write $A(u)$ for the membership of u in A , or, using lambda notation for denoting functions, we write the fuzzy set A itself as

$$A = \lambda u. A(u)$$

If a variable v has type τ_1 and an expression e has type τ_2 , then $\lambda v. e$ has type $\tau_1 \rightarrow \tau_2$, so A above has type $U \rightarrow [0,1]$. We will be using function types of the form $U \rightarrow [0,1]$ often to represent fuzzy sets, and so we abbreviate this type as “ ϕ_U ”. We use “ $a : \tau$ ” to state that object a has type τ . Thus A is concisely described as “ $A : \phi_U$ ”, and $A(u)$ as “ $A(u) : [0,1]$ ”.

3. TRIANGULAR NORMS, CONORMS, AND NEGATIONS

Fuzzy logic values (in $[0,1]$) may be combined in ways analogous to the logical operations on crisp logic values ($\{true, false\}$). Without specifying the operators exactly, we may describe the possible fuzzy versions of conjunction, disjunction, and negation by abstractly specifying their properties. It happens that conjunctions are suitably represented by triangular norms, and disjunctions are suitably represented by triangular conorms, as follows.

A *triangular norm* (abbreviated *t-norm*) [9] is a function $T : [0,1] \times [0,1] \rightarrow [0,1]$ which fulfills

- (1) $T(0, 0) = 0,$ *(Bottom stability)*
- (2) $T(u, 1) = u,$ *(Top identity)*
- (3) $u \leq u' \text{ and } v \leq v' \Rightarrow T(u, v) \leq T(u', v'),$ *(Monotonicity)*
- (4) $T(u, v) = T(v, u),$ *(Commutativity)*
- (5) $T(T(u, v), w) = T(u, T(v, w)).$ *(Associativity)*

A *triangular conorm* (abbreviated *t-conorm*) is a function $S : [0,1] \times [0,1] \rightarrow [0,1]$ which fulfills

- (6) $S(1, 1) = 1,$ *(Top stability)*
- (7) $S(u, 0) = u,$ *(Bottom identity)*

as well as the monotonicity (3), commutativity (4), and associativity (5) conditions above for t-norms.

A *negation* is a function $N : [0,1] \rightarrow [0,1]$ which fulfills

- (8) $u \leq u' \Rightarrow N(u) \geq N(u'),$ *(Monotonicity)*
- (9) $N(N(u)) = u.$ *(Involution)*

Some of the most interesting t-norms are listed by Klement [9] as

$$T_s(u, v) = \begin{cases} \min(u, v) & \text{if } s = 0 \\ uv & \text{if } s = 1 \\ \max(u + v - 1, 0) & \text{if } s = \infty \\ \log_s \left(1 + \frac{(s^u - 1)(s^v - 1)}{s - 1} \right) & \text{otherwise} \end{cases}$$

$$T_w(u, v) = \begin{cases} \min(u, v) & \text{if } \max(u, v) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$T^\alpha(u, v) = \frac{uv}{\alpha + (1 - \alpha)(u + v - uv)} \quad (\alpha \geq 0)$$

These t-norms have dual t-conorms, as follows:

$$S_s(u, v) = \begin{cases} \max(u, v) & \text{if } s = 0 \\ u + v - uv & \text{if } s = 1 \\ \min(u + v, 1) & \text{if } s = \infty \\ 1 - \log_s \left(1 + \frac{(s^{1-u} - 1)(s^{1-v} - 1)}{s - 1} \right) & \text{otherwise} \end{cases}$$

$$S_w(u, v) = \begin{cases} \max(u, v) & \text{if } \min(u, v) = 0 \\ 1 & \text{otherwise} \end{cases}$$

$$S^\beta(u, v) = \frac{u + v + (\beta - 1)uv}{1 + \beta uv} \quad (\beta \geq -1)$$

The most interesting negations are:

$$N^\gamma(u) = \frac{1-u}{1+\gamma u} \quad (\gamma > -1)$$

In combination, we also want these operators to satisfy DeMorgan's laws:

$$(10) \quad N(S(u, v)) = T(N(u), N(v)) \quad (\text{DeMorgan's Law 1})$$

$$(11) \quad N(T(u, v)) = S(N(u), N(v)) \quad (\text{DeMorgan's Law 2})$$

Examples of (T, S, N) triples of triangular norms, conorms, and negations that satisfy these DeMorgan's laws are

$$(T_s, S_s, N^0) \quad \text{for } s \in [0, \infty]$$

$$(T_0, S_0, N^\gamma) \quad \text{for } \gamma > -1$$

$$(T_w, S_w, N^0)$$

$$(T^\alpha, S^\beta, N^\gamma) \quad \text{for } \alpha \geq 0, \beta \geq -1, \gamma > -1, \text{ and } \alpha = \frac{1+\beta}{1+\gamma}$$

These include many of the definitions of fuzzy conjunction, disjunction, and negation that have been proposed. The most common definitions used for conjunction and disjunction are minimum (T_0) and maximum (S_0) respectively. The most common definition for negation is $1-u$ (N^0). Some writers avoid the use of T_1 and S_1 , as these suggest a probabilistic semantics, and there is a desire to distinguish fuzzy calculations from probability. We will be using these families of functions, especially triangular norms, to aid in defining other operators later.

4. CLASSICAL FUZZY INFERENCE

When calculating according to rules in a fuzzy expert system, the central operation is called "inferencing," and is concerned with taking rules of the knowledge base, applying known information, and deriving new information. In [10], the standard problem of fuzzy inferencing is illustrated by the following:

$$\begin{array}{ll} \text{given} & \text{if } X \text{ is } A \text{ then } Y \text{ is } B \\ \text{and} & \underline{X \text{ is } A'} \\ \text{conclude} & Y \text{ is } B' \end{array}$$

Here A' and B' are fuzzy sets related to but not necessarily equal to A and B , respectively. This is known as *compositional modus ponens*, and can be expressed in classical fuzzy logic as

$$B' = A' \circ (A \Rightarrow B)$$

where \Rightarrow is a fuzzy implication operator, and \circ is a fuzzy max-* composition operator. Commonly, a fuzzy expert system consists of a number of rules that cooperate in forming an answer, as in

$$\begin{array}{ll} \text{Rule 1.} & \text{If } X \text{ is } A_1 \text{ then } Y \text{ is } B_1 \\ \text{Rule 2.} & \text{Else If } X \text{ is } A_2 \text{ then } Y \text{ is } B_2 \\ \vdots & \vdots \\ \text{Rule n.} & \text{Else If } X \text{ is } A_n \text{ then } Y \text{ is } B_n \end{array}$$

When these rules are applied to the fact X is A' , they yield the resulting fuzzy sets B_1', \dots, B_n' . The result of the entire fuzzy expert system is then the combination of these fuzzy sets using an operator f_{else} , so the result set B' is

$$B' = B_1' f_{else} \dots f_{else} B_n'$$

These three operators define the classical inferencing process. They are now described in more detail in the next three sections, first \Rightarrow , then \circ , and finally f_{else} .

4.1. Fuzzy implication

There is a vast multitude of suggestions for fuzzy implication. In normal set theory, a relation R between sets A and B is a subset of $A \times B$. In fuzzy logic, if U and V are the universes of discourse of fuzzy sets A and B respectively, then a fuzzy relation R between fuzzy sets A and B is a fuzzy set based on the universe of discourse

$U \times V$. Fuzzy implication $\Rightarrow : \phi_U \times \phi_V \rightarrow \phi_{U \times V}$ is an operation which yields such a fuzzy relation $R : \phi_{U \times V}$, and we write $R = A \Rightarrow B$. If the relation R is independent of the actual elements of U and V , and only depends on their degrees of membership in A and B , then a suitable function $r : [0,1] \times [0,1] \rightarrow [0,1]$ exists so that R can be defined as the pointwise application of r to the cross product $A \times B$, as

$$R = A \Rightarrow B = \lambda u,v. r(A(u), B(v))$$

There is extensive debate in the literature about the particular operation to use for r . Mizumoto lists fifteen different operators that have been proposed for fuzzy implication [12]. Bouchon-Meunier lists eight implication operators, and in addition two families of operators, parameterized by functions [2]. The operators she gives for r are (using Mizumoto's subscripts)

$$\begin{aligned} r_m &= \lambda a,b. \max(1-a, \min(a, b)) && \text{(Willmott implication)} \\ r_a &= \lambda a,b. \min(1-a+b, 1) && \text{(Lukasiewicz implication)} \\ r_c &= \lambda a,b. \min(a, b) && \text{(Mamdani implication)} \\ r_b &= \lambda a,b. \max(1-a, b) && \text{(Kleene-Dienes implication)} \\ r_* &= \lambda a,b. 1-a+ab && \text{(Reichenbach implication)} \\ r_s &= \lambda a,b. \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases} && \text{(Rescher-Gaines implication)} \\ r_g &= \lambda a,b. \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases} && \text{(Brouwer-Gödel implication)} \\ r_\Delta &= \lambda a,b. \begin{cases} 1 & \text{if } a \leq b \\ b/a & \text{otherwise} \end{cases} && \text{(Goguen implication)} \end{aligned}$$

One family of operators Bouchon-Meunier gives is

$$r_N = \lambda a,b. \max(N(a), b)$$

where $N : [0,1] \rightarrow [0,1]$ is a *strong negation* defined by

$$N(x) = g^{(-1)}(g(0) - g(x))$$

where $g : [0,1] \rightarrow \mathbb{R}^+$ is a continuous, strictly decreasing function defined on $[0,1]$ and lying on \mathbb{R}^+ , so that $g(1) = 0$ and $g(0) < +\infty$, and where

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \text{if } 0 \leq x \leq g(0) \\ 0 & \text{if } x > g(0) \\ 1 & \text{if } x < 0. \end{cases}$$

As an example, for $g(x) = 1 - x$, we find that $g^{-1}(x) = 1 - x$, and derive $N(x) = 1 - x$, which is the negation N^0 , and from this we get $r_N = \lambda a,b. \max(1-a, b)$, which is the implication r_b .

Such a function g is an *additive generator* of a triangular norm F , where we define F by

$$F(x, y) = g^{(-1)}(g(x) + g(y))$$

Note that for $g(x) = 1 - x$, we derive $F(x, y) = \max(0, x + y - 1)$, which is the triangular norm T_∞ , and for $g(x) = -\ln x$, we derive $F(x, y) = xy$, which is the triangular norm T_1 .

The other family of operators Bouchon-Meunier gives is

$$r_F = \lambda a,b. f(a, b)$$

where $f : [0,1] \times [0,1] \rightarrow [0,1]$ is the *quasi-inverse* of a triangular norm F , defined by

$$f(u, v) = \max\{w \in [0, 1] \text{ such that } F(u, w) \leq v\} \quad (\text{defn. of quasi-inverse})$$

For $F = T_0 = \lambda a, b. \min(a, b)$, the quasi-inverse is $f(a, b) = 1$ if $a \leq b$, and b otherwise, which is r_g . For $F = T_1 = \lambda a, b. ab$, the quasi-inverse is $f(a, b) = 1$ if $a \leq b$, and b/a otherwise, which gives r_Δ . For $F = T_\infty = \lambda a, b. \max(a + b - 1, 0)$, the quasi-inverse is $f(a, b) = \min(1 - a + b, 1)$, yielding r_a .

Clearly there is an enormous diversity of choices for the implication operator. For each r , there is a corresponding R and \Rightarrow , as described above. In addition to the variety, there is debate about their individual suitability. Notice, for example, that r_c does not even match the crisp implication operator \Rightarrow for the cases where $a = 0$ (false):

$$r_c(a, b) = \min(a, b) = \min(0, b) = 0$$

whereas in classical, crisp logic, using $0 = \text{false}$ and $1 = \text{true}$, $(0 \Rightarrow b) = 1$. Bouchon-Meunier remarks that r_c “does not satisfy the same properties as the other ones and cannot be considered as a real fuzzy implication.” [2] Nevertheless, this has not prevented the use of r_c in many papers and implementations, such as [10] and Togai’s work.

4.2. Fuzzy max-* composition

The fuzzy max-* composition operator \circ has type $\phi_U \times \phi_{U \times V} \rightarrow \phi_V$. If $C = (A \Rightarrow B)$, then

$$A' \circ (A \Rightarrow B) = A' \circ C = \lambda v. \bigcup_{u \in U} (A'(u) * C(u, v))$$

where $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ specifies the composition operator. A triangular norm is classically considered for $*$ [2]. The most common choice for the $*$ operator is minimum (T_0), producing Zadeh’s max-min composition. Other compositional operators have been suggested, for example, Dubois and Prade have respectively suggested the bounded product Θ and drastic product Λ , equivalent to the triangular norms $T_\infty = \lambda u, v. \max(u + v - 1, 0)$ and $T_w = \lambda u, v. (\min(u, v) \text{ if } \max(u, v) = 1, \text{ else } 0)$, which generate the max- Θ and max- Λ composition operators.

4.3. Alternative combination

The fuzzy alternative combination operator f_{else} is used to combine the results of different rules without priority. f_{else} has type $\phi_V \times \phi_V \rightarrow \phi_V$. Clearly f_{else} should be both commutative and associative. Several definitions of f_{else} are discussed in [10]. Two possible definitions are the or-link (f_{or}) and the and-link (f_{and}):

$$\begin{aligned} B_1' f_{or} B_2' &= \max(B_1', B_2') &= \lambda v. \max(B_1'(v), B_2'(v)) \\ B_1' f_{and} B_2' &= \min(B_1', B_2') &= \lambda v. \min(B_1'(v), B_2'(v)) \end{aligned}$$

The f_{or} operator models the intuition that all the rules describe alternative situations, and that the final conclusion is to be the aggregation of all the results of these rules. This operator monotonically increases the result for each constituent. This considers the different rules as specifying different responses to various cases, with each rule that fires contributing positively to the eventual effect.

The f_{and} operator models the intuition that all the rules describe competing situations, and that the final conclusion is to be the most skeptical value of all the results of these rules. This operator monotonically decreases the result for each constituent. This appears to be less intuitive than f_{or} , but it in fact is necessary for certain choices of the other operators. In fact, the use of most of the implication operators (except r_c) imply that rules which are not at all applicable to the current state of the world yield complete fuzzy sets ($B'(v) = 1.0$ for all v). If these are combined with other resulting fuzzy sets using f_{or} , they swamp out the contributions from other rules to yield complete fuzzy sets, lacking relevance to the situation. To avoid this, Dubois and Prade suggest using the f_{and} operation for combination when r_m is used for implication [5], and this works for most of the other implication operators as well. By contrast, f_{or} works appropriately with r_c .

In classical fuzzy inferencing, the choice of the f_{else} operator is dependent on the context of the rules and the type of reasoning desired, as well as depending on the other operators. [2]

4.4. Review of the classical approach

In the great diversity of proposals in formulating fuzzy reasoning, we see a complex, interdependent system of operators, without clear preferences, i.e., muddled. There are so many ways to construct the inferencing operators, all apparently reasonable yet inducing hidden dependencies. In general, when a situation becomes muddled, the solution is to draw back and reconsider the assumptions underlying the model.

5. TOGAI'S METHODS OF FUZZY INFERENCE

Togai InfraLogic, Inc. is a significant, successful, and well-known source of practical, working fuzzy expert systems. Togai uses two forms of fuzzy inferencing, which they term “max-min” and “max-dot” (or “max-product”) inferencing [8], succinctly described by:

$$\begin{aligned} \text{Max-Min: } B' &= \lambda v. \min\left(\bigcup_{u \in U} \min(A'(u), A(u)), B(v)\right) = (\bigcup (A' \cap A)) \cap B \\ \text{Max-Dot: } B' &= \lambda v. \left(\bigcup_{u \in U} \min(A'(u), A(u))\right) \cdot B(v) = (\bigcup (A' \cap A)) \cdot B \end{aligned}$$

Togai first combines A' and A , and then combines the result with B . We believe that it is inappropriate to describe Togai's methods in the classical formulation, because that is not how they are computed. Their process is different, and the underlying cause is that the classical formulation is an extension of crisp boolean logic, whereas expert systems are not based on crisp boolean logic. In particular, expert system rules are not logical implications.

While at first it may seem a reasonable extension to base the fuzzy inferencing operators on the classical implication operator and modus ponens, *in fact this does not reflect the typical operation of an expert system*. In a normal, crisp expert system, rules are not processed according to classical implication and modus ponens. Rather, in the simplest formulation, rules are processed in a loop called the “recognize-act cycle” [16]. Once each cycle, each rule's antecedent is examined and tested to see if it is true. All rules whose antecedent tested true are collected into a set called the “activation set”. According to a pre-set “conflict resolution” strategy [11], one of these rules is chosen from the activation set and its consequent is executed. This is called “firing” the rule. Then the cycle repeats. In practice, the process of finding the activated rules is highly optimized using the Rete algorithm [6], [7].

This processing does not regard the antecedent and consequent parts of a rule as merged into an homogeneous implication relation; rather it is more of the style of an “if” statement in a programming language, where if the test is true, the body may then be executed; as evidence of this, the test and the body are different *types* of phrases. Also, the test and the body are separated in time and causality; the antecedent's test of relevance is the first operation that happens, and only secondly and as a result is the consequent empowered to operate. An expert system's rule compares to a logical implication as in C, assignment (=) compares to the equality test (==).

The max-min and max-dot methods also are the inferencing methods most used in other practical work besides Togai's. One reason may be that forming a fuzzy relation among the clauses of the antecedent of the rule and the consequent is complex, and difficult and costly to implement. General articles surveying the field demonstrate the max-min method [3, 4], sometimes only considering crisp values for A' .

Togai's max-min method can possibly be expressed in the classical framework by making the adroit choices of \Rightarrow_c (Mamdani implication) for \Rightarrow and max-min composition for \circ [15], but this does not straightforwardly describe the essence of this method. The other method, max-dot, cannot be expressed in the classical formulation, a proof of which is presented in Appendix A. Nevertheless, these methods of Togai work to produce real, practical expert systems. We suggest that expert systems deserve a fuzzy theoretical framework appropriate for that field.

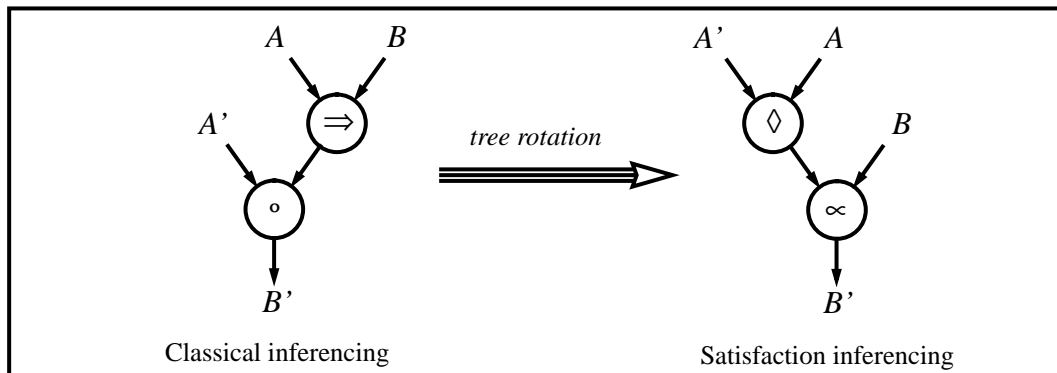
6. SATISFACTION-ORIENTED FUZZY INFERENCE

These considerations motivate us to propose a simple but fundamentally different formulation of fuzzy inferencing, *rotating* the operators. Instead of the classical formulation,

$$B' = A' \circ (A \Rightarrow B)$$

we propose

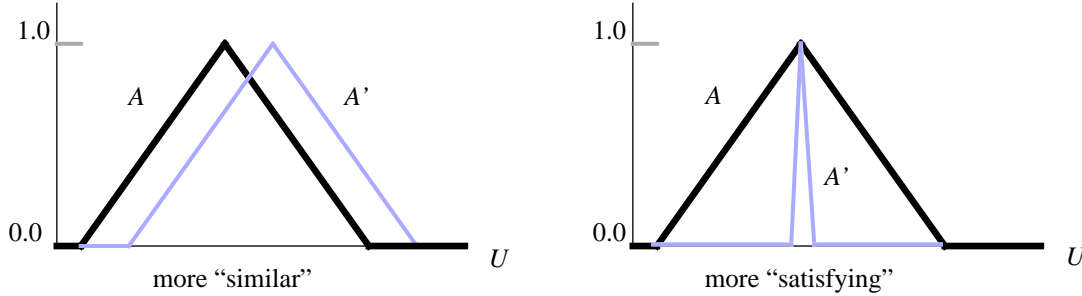
$$B' = (A' \diamond A) \infty B$$



This describes applying a rule to a specific case by first comparing the condition of the rule to the actual facts, and then if they match, accepting the conclusion of the rule, qualified by the success of the match. Here $\diamond : \phi_U \times \phi_U \rightarrow [0,1]$ yields the degree of *satisfaction* between A and A' , and $\infty : [0,1] \times \phi_V \rightarrow \phi_V$ is a *muting* operator, modifying the degree of membership in B according to the degree of satisfaction. We call this *satisfaction-oriented fuzzy inferencing*.

6.1. Satisfaction operator

In defining \diamond , it is tempting to immediately choose a measure of “similarity” between the two fuzzy sets A and A' ; after all, they are both based on the same universe of discourse; if they are exactly equal, one would expect the measure should be 1.0; and if they are completely disjoint (no non-zero memberships in common), one would expect the measure should be 0.0.



However, take for example A defined as in the solid line in the figure above, then the A' given in the left example is clearly more “similar” to A than the A' in the right example. The area of overlap is large, while the area of difference between the two curves is small. Yet the right example represents an A' which is a fuzzy set version of a crisp value, where the crisp value coincides with the maximum of A . To be consistent with the direct use of the crisp value, the right example should also give the maximum measure of 1.0.

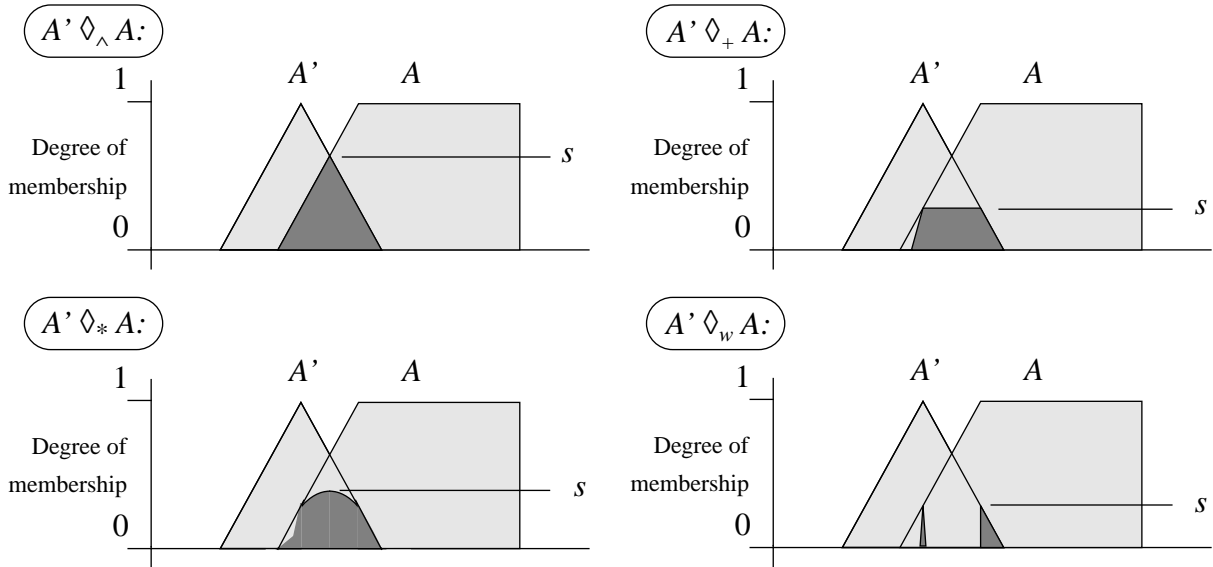
The problem is that we have been looking at the two fuzzy sets A and A' and considering their closeness to equality; but when we apply a crisp *value* to a fuzzy set, we do not consider *their* closeness (since clearly a point is not a set), but rather the degree to which the crisp value satisfies the membership test. In the same fashion, when we compare A and A' , we should consider to what degree they *satisfy* each other. In this sense, we regard each fuzzy set as a test for the other. The symmetry inherent in this description implies a symmetry between the fuzzy set operands in the definition of the satisfaction operator.

We define the *max-** satisfaction family of operators by

$$A' \diamond A = \bigcup_{u \in U} (A'(u) * A(u))$$

where $*$ has type $[0,1] \times [0,1] \rightarrow [0,1]$. Similar to the choices for the composition operator, we suggest a triangular norm for $*$. Choosing minimum (T_0) as $*$ forms the max-min satisfaction operator. Note the similarity of this to the inner part of the definition of Zadeh’s max-min composition operator. Analogously, we may define max- Θ and max- Λ satisfaction operators in accord with the max- Θ and max- Λ composition operators, where the $*$ operator in the definition above is replaced by the bounded product Θ (T_∞) or the drastic product Λ (T_w). These also support the intuition of satisfaction rather than similarity. Four examples, with their triangular norms, are:

$$\begin{aligned} A' \diamond_{\wedge} A &= \bigcup_{u \in U} \min(A'(u), A(u)) && (\text{max-min satisfaction}) && (T_0) \\ A' \diamond_{\cdot} A &= \bigcup_{u \in U} A'(u) \cdot A(u) && (\text{max-dot satisfaction}) && (T_1) \\ A' \diamond_{+} A &= \bigcup_{u \in U} \max(A'(u) + A(u) - 1, 0) && (\text{max-}\Theta \text{ satisfaction}) && (T_\infty) \\ A' \diamond_w A &= \bigcup_{u \in U} \begin{cases} \min(A'(u), A(u)) & \text{if } \max(A'(u), A(u)) = 1 \\ 0 & \text{otherwise} \end{cases} && (\text{max-}\Lambda \text{ satisfaction}) && (T_w) \end{aligned}$$

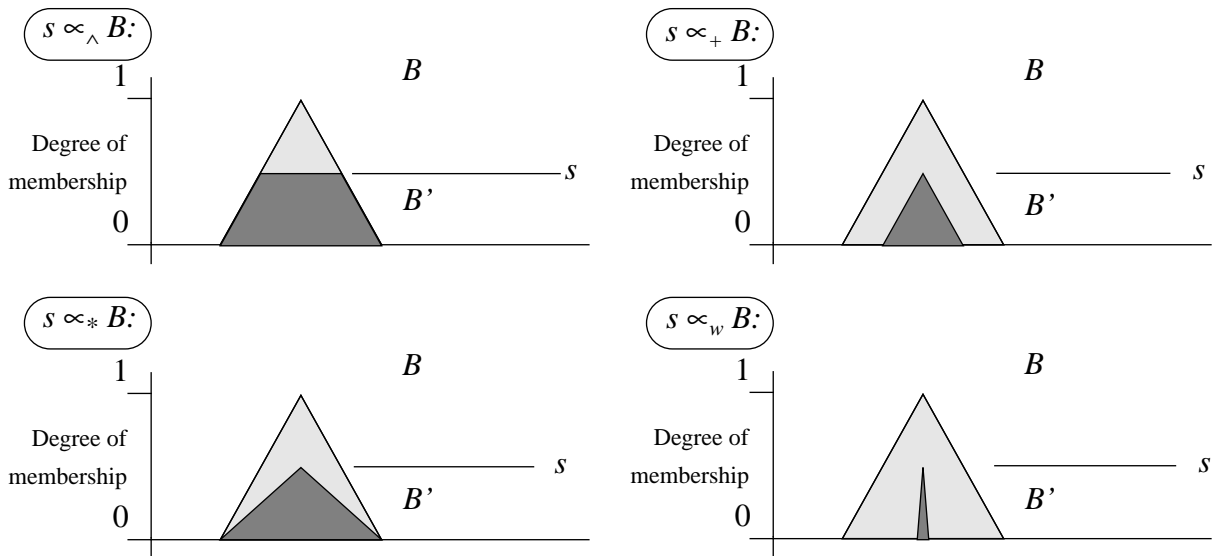


6.2. Muting operator

The muting operator ∞ has type $[0,1] \times \phi_V \rightarrow \phi_V$. The purpose of $s \infty B$ is to produce a resulting fuzzy set B' which is like B but possibly lessened in some way if $s < 1$. We suggest $s \infty B = \lambda v. s * B(v)$, where $*$ is a triangular norm.

Four simple muting operators, with their triangular norms, are:

$$\begin{aligned}
 s \infty_{\wedge} B &= \lambda v. \min(s, B(v)) && \text{(leveled muting)} && (T_0) \\
 s \infty_{*} B &= \lambda v. s \cdot B(v) && \text{(scaled muting)} && (T_1) \\
 s \infty_{+} B &= \lambda v. \max(B(v) + s - 1, 0) && \text{(lowered muting)} && (T_{\infty}) \\
 s \infty_w B &= \lambda v. \begin{cases} \min(s, B(v)) & \text{if } \max(s, B(v)) = 1 \\ 0 & \text{otherwise} \end{cases} && \text{(drastic muting)} && (T_w)
 \end{aligned}$$



There is no correspondence between the muting operator ∞ and the implication operator \Rightarrow from the classical formulation, as there was between the similarity operator \diamond and the detachment operator \circ . Therefore the diversity present in the suggestions for \Rightarrow is not transferred to ∞ .

6.3. Alternative combination

Any of the triangular conorms from section 3 could be used for f_{else} , but f_{or} (S_0) seems the most natural choice for satisfaction-oriented inferencing. There is no need to avoid using f_{or} as in the classical approach, as rules that do not apply do not contribute strong fuzzy sets. This supports the positive intuition of aggregating contributions. It also coincides with the normal expert system interpretation of a set of rules as being disjunctive cases.

6.4. Comparison with Togai's inferencing methods

Essentially, the satisfaction approach is a generalization of the inferencing presented by Togai. Togai's two inferencing methods, "max-min" and "max-dot" (or "max-product"), are both expressed naturally within satisfaction-oriented fuzzy inferencing, using the max-min satisfaction operator, \diamond_{\wedge} . The "max-min" method uses ∞_{\wedge} as the muting operator, and the "max-dot" method uses ∞_* as the muting operator. Togai uses f_{or} for the combination operator, as we suggest as well for satisfaction-oriented inferencing. The notation presented here to describe satisfaction-oriented fuzzy reasoning, using \diamond and ∞ , is new and not used by Togai. The two methods can now be succinctly written as

$$\begin{aligned} \text{Max-Min:} \quad B' &= (A' \diamond_{\wedge} A) \infty_{\wedge} B &= \lambda v. \min\left(\bigcup_{u \in U} \min(A'(u), A(u)), B(v)\right) \\ \text{Max-Dot:} \quad B' &= (A' \diamond_{\wedge} A) \infty_* B &= \lambda v. \left(\bigcup_{u \in U} \min(A'(u), A(u))\right) \cdot B(v) \end{aligned}$$

7. COMPARISON OF THE CLASSICAL AND SATISFACTION APPROACHES

The types of the operators used are different in the two approaches:

Classical:

$$\begin{aligned} \Rightarrow : & \quad \phi_U \times \phi_V \rightarrow \phi_{U \times V} \\ \circ : & \quad \phi_U \times \phi_{U \times V} \rightarrow \phi_V \end{aligned}$$

Satisfaction:

$$\begin{aligned} \diamond : & \quad \phi_U \times \phi_U \rightarrow [0,1] \\ \infty : & \quad [0,1] \times \phi_V \rightarrow \phi_V \end{aligned}$$

Both:

$$f_{else} : \quad \phi_V \times \phi_V \rightarrow \phi_V$$

The satisfaction approach is objectively simpler, due to the fact that the types involved in the satisfaction-oriented formulation are less complex types, of lower order. In particular, the type of the intermediate value produced by the first operator and passed to the second as an argument is far less complex ($\phi_{U \times V}$ vs. $[0,1]$). This implies less design freedom in specifying the satisfaction operators, but also better guidance in choosing natural definitions of the operators, focussing research effort. The example that will be presented will show the simplicity of this approach.

The intermediate value of the classical approach is a complex data object representing a fuzzy relation. This is relatively difficult to grasp intuitively. If the antecedent of a rule has one clause, this relation has a three-dimensional graph. If the antecedent has n clauses, the graph has $n+2$ dimensions. By contrast, the intermediate value in the satisfaction method is a real number between 0 and 1. This is easily interpreted as the degree to which the test of the antecedent succeeded. The way that this value is then used to qualify the conclusion is easily grasped and more intuitive than the classical composition operator's method of extracting a fuzzy set from the fuzzy implication relation. This will also be demonstrated in the example.

As far as ease of implementation, the simplicity alone eases this. Besides this, virtually all practical implementations of fuzzy expert systems use essentially a simple version of the satisfaction approach, not only Togai. Although they may recognize there is a difference between what they are doing and the classical theory, they have found the satisfaction method more implementable, even at the cost of not having a theoretical basis for their systems.

In terms of speed of execution, the classical method seems to have an advantage in that the fuzzy implication relation may be precomputed for each rule, and then at runtime only the composition operator need be applied. However, it appears that this does not necessarily preclude the satisfaction approach being faster; computing the satisfaction operator (\diamond) should take less time than the composition operator (\circ), perhaps much less time, and then using the satisfaction value to mute the conclusion (∞) should take less time than the satisfaction operation.

Since both approaches take as input a full fuzzy set and yield as output a full fuzzy set, both approaches are equally able to produce chains of inference.

In the classical approach, the operators are interdependent; Bouchon-Meunier reports that a “careful” combined choice of the appropriate implication, composition, and combination operators “is necessary to avoid unacceptable conclusions” [2]. By contrast, the satisfaction operators are independent, due to the simple type of the intermediate value passed between them ([0,1]). In terms of software engineering principles, this provides the least possible coupling between the operators, and thus the highest degree of modularity [13].

The classical approach describes a static relationship between the fuzzy sets in a rule’s antecedent and consequent. This is a declaration of a logical dependency, related to the implication operator in classical logic. It is not active or operational. This does not make it *wrong*, as long as the designer of the rules understands the semantics being used, but it is inconsistent with the semantics of crisp expert systems.

The satisfaction-oriented approach specifies a dynamic, operational execution of a rule, not a static relationship between fuzzy sets. This matches the process of firing crisp rules in an expert system, and also our intuitive feelings of causality, that *if* the test of the rule is satisfied, *then* the body is executed and the consequent asserted. This approach eases the unification of crisp and fuzzy rules into one concept, and the integration of fuzzy and crisp reasoning in a single expert system. We will discuss our approach to this integration in an upcoming paper.

There is a point of intersection between the two approaches. If we choose \Rightarrow_c (Mamdani implication) for \Rightarrow , max-min composition for \circ , max-min satisfaction for \diamond , and ∞_{\wedge} (leveled muting) for ∞ , then the two inference methods give the same result:

Classical:

$$B' = A' \circ (A \Rightarrow B) = \lambda v. \bigcup_{u \in U} \min(A'(u), \min(A(u), B(v)))$$

Satisfaction:

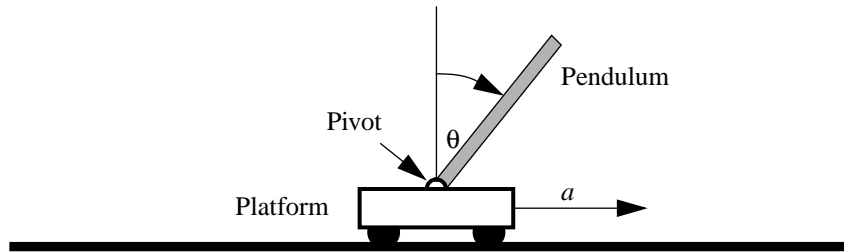
$$B' = (A' \diamond A) \infty B = \lambda v. \min((\bigcup_{u \in U} \min(A'(u), A(u))), B(v))$$

Clearly both are equal to

$$B' = \lambda v. \bigcup_{u \in U} \min(A'(u), A(u), B(v))$$

8. EXAMPLE

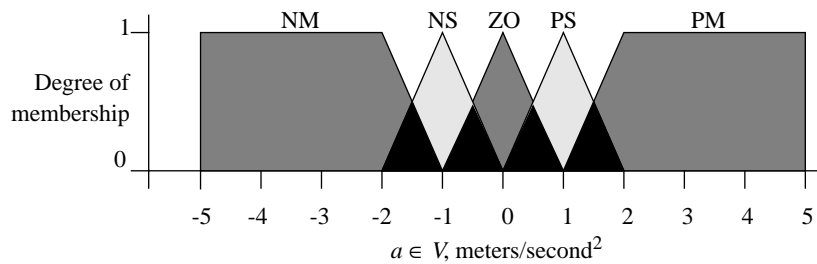
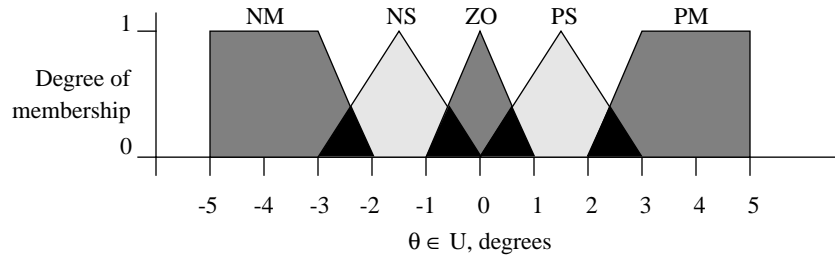
We will show that the satisfaction approach reflects one’s natural physical intuition, whereas the classical approach does not. The following is a very simple version of the canonical example. Consider an inverted pendulum attached to a pivot on a movable platform, where we can control the acceleration of the platform left and right, with the objective of keeping the pendulum balanced if it begins to fall.



Let $\theta \in U = [-5.0, 5.0]$ denote the current angle of the pendulum in degrees, with positive angles denoting clockwise from upright. Let $a \in V = [-5.0, 5.0]$ denote the current acceleration of the platform in m/sec^2 , with positive accelerations to the right.

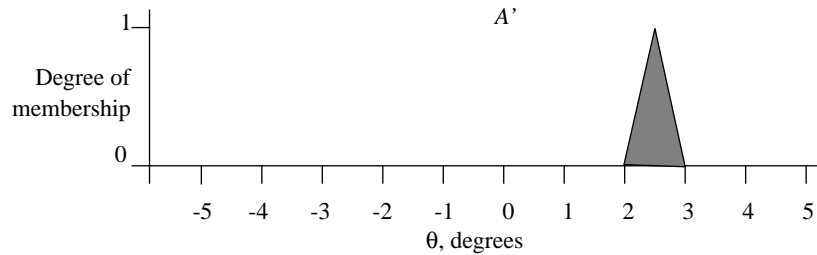
8.1. Membership functions

Let the membership functions be as indicated in the following diagrams:



The graphs above show the membership functions of characteristic fuzzy sets defined on the two universes of discourse. For example, the V universe of discourse has five fuzzy sets defined, named NM, NS, ZO, PS, and PM, standing for, respectively, “Negative Medium,” “Negative Small,” “Zero,” “Positive Small,” and “Positive Medium.” The graph under the name “NM” states that the membership function $NM(a)$ is 1.0 for a between -5 and -2, decreases linearly from 1.0 to 0.0 for a between -2 and -1, and is 0.0 for $a \geq -1$. The names of the fuzzy sets are reused across different universes, but this overloading is unambiguous as long as the universe is known.

Take the case where θ is “about” 2.5 degrees. Let A' have the possibility distribution



8.2. Rule base

Assume that the rule base has the following five rules:

- (R1) **if** θ is PM
then a is PM
- (R2) **if** θ is PS
then a is PS
- (R3) **if** θ is ZO
then a is ZO
- (R4) **if** θ is NS
then a is NS
- (R5) **if** θ is NM
then a is NM

RULES:

	θ is NM	θ is NS	θ is ZO	θ is PS	θ is PM
	R5: a is NM	R4: a is NS	R3: a is ZO	R2: a is PS	R1: a is PM

This example is intentionally as simple and direct as possible. One could almost have just computed the result a from the input θ as simply $a = \theta$. The reason for this simple system is not to show the versatility of expert systems (that is assumed), but to examine in a clean experiment the different kinds of fuzzy inferencing.

The purpose of this example is to demonstrate the calculations involved in each style of inferencing, to show simplicity and intuitiveness, not to make a general conclusion about their relative validity. The exact semantics of a fuzzy expert system depends on the triple of the membership functions, the rules, and the inferencing method, and thus the two systems cannot be completely compared in any one example.

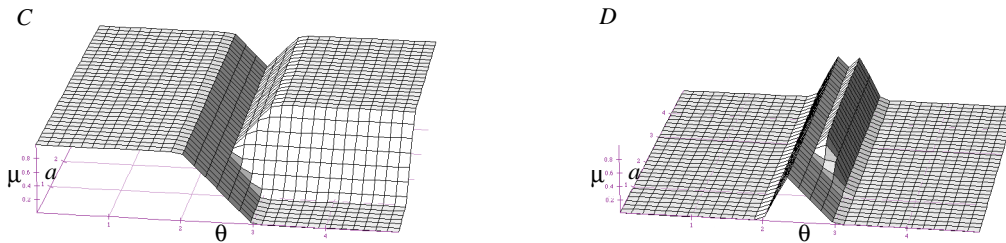
8.3. Classical inferencing

Among the various choices for \Rightarrow and \circ , in this example we will use \Rightarrow_m and max-min composition. For this value of A' , most of the rules will not contribute meaningfully to the final answer, but rule 1 is one that will. For rule 1, let $A = \text{"}\theta \text{ is PM"}$, and $B = \text{"}a \text{ is PM"}$.

Then using \Rightarrow_m for \Rightarrow , we calculate $C = A \Rightarrow B$ as

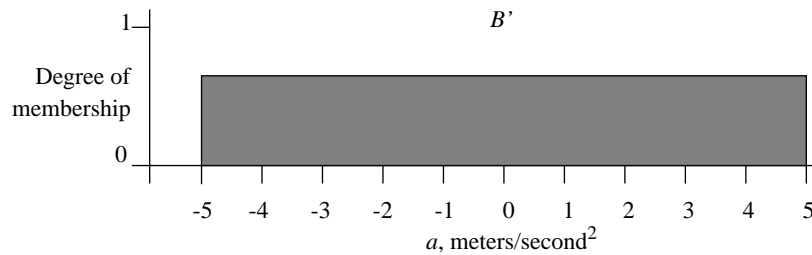
$$C = \lambda\theta, a. \max(\min(A(\theta), B(a)), 1 - A(\theta))$$

This fuzzy relation C is a relatively complex intermediate value. Its graph is below, showing membership values (μ) of C for positive θ, a (the rest is a continuation).



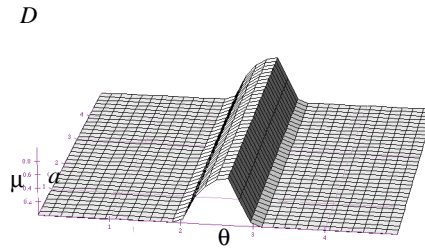
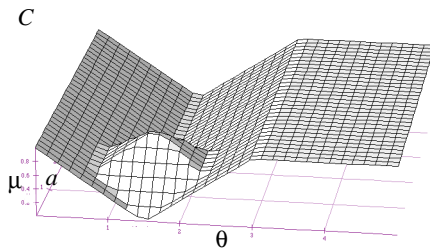
We then combine C with A' to form $D = \lambda\theta, a. \min(A'(\theta), C(a))$ as shown above. Taking the maximum of D over all θ , we arrive at the resultant contribution of rule 1 to the value of B' .

$$\begin{aligned} B' &= A' \circ (A \Rightarrow B) = A' \circ C \\ &= \lambda a. \bigcup_{\theta \in U} \min(A'(\theta), C(\theta, a)) \\ &= \lambda a. \bigcup_{\theta \in U} D(\theta, a) \\ &= \lambda a. 0.67 \end{aligned}$$



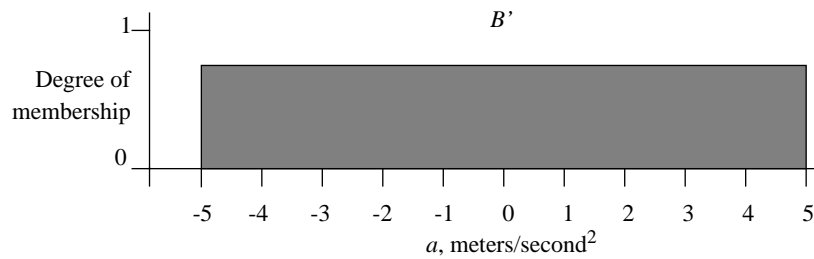
Applying Rule 2:

The only other rule to apply to this state of the world, where θ is about 2.5, is rule 2. For this rule, $A = \text{"}\theta \text{ is PS"}$, and $B = \text{"}a \text{ is PS"}$, and of course these new A, B generate a new fuzzy implication relation C and composition relation D .



Then the fuzzy set yielded by rule 2 is

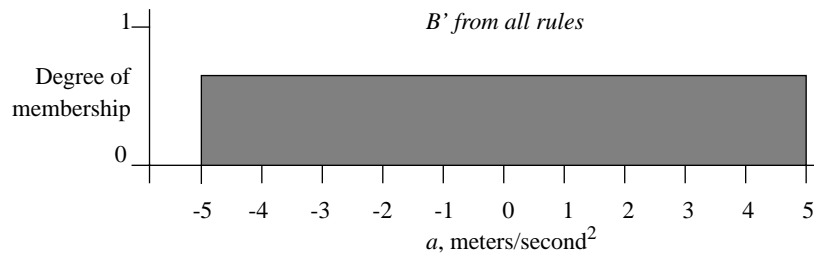
$$\begin{aligned}
 B' &= A' \circ (A \Rightarrow B) = A' \circ C \\
 &= \lambda a. \bigcup_{\theta \in U} \min(A'(\theta), C(\theta, a)) \\
 &= \lambda a. \bigcup_{\theta \in U} D(\theta, a) \\
 &= \lambda a. 0.75
 \end{aligned}$$



Combining the Rules' Results:

Then combining these two fuzzy sets obtained from rules 1 and 2 with f_{and} (as suggested by Dubois and Prade [5]), we find the final output of the fuzzy expert system to be the minimum:

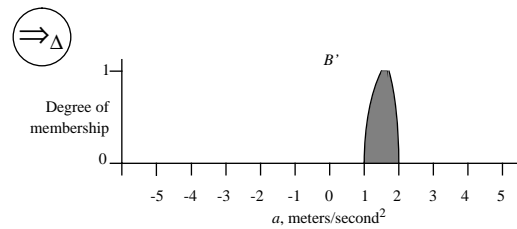
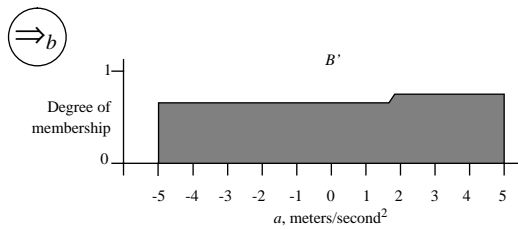
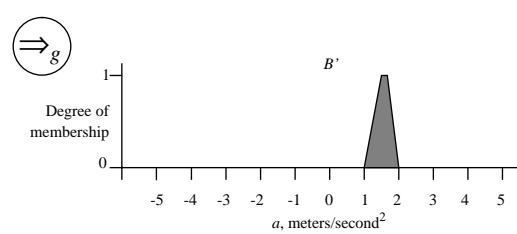
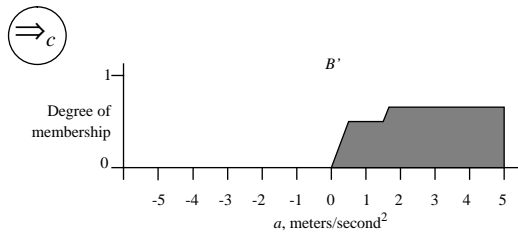
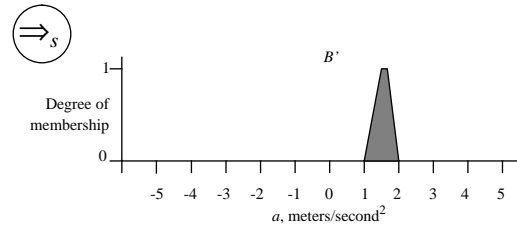
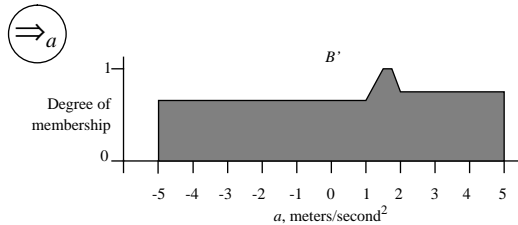
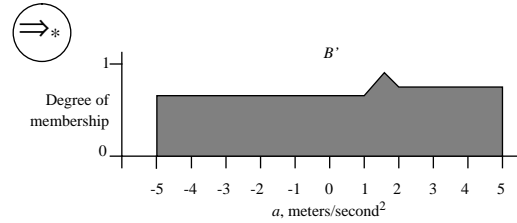
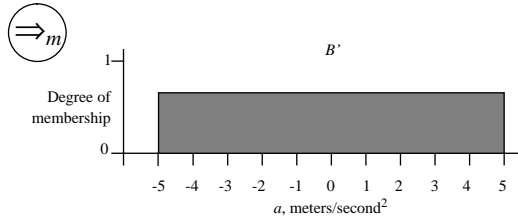
$$B' = \lambda a. (0.67) f_{and} \lambda a. (0.75) = \lambda a. \min(0.67, 0.75) = \lambda a. 0.67$$



Thus the result of all this complex calculation is a flat fuzzy set with no preference for any particular acceleration. *This will not balance the inverted pendulum*, which in this example is tipped forward and falling forward.

But is this simply characteristic of the particular implication operator chosen? The next set of graphs show the resulting B' fuzzy sets for eight different choices of the fuzzy implication operator \Rightarrow , using max-min composition and f_{and} (except for \Rightarrow_c , for which we must use f_{or} to avoid a trivially empty result).

Most of the other choices for fuzzy implication are not significantly better than \Rightarrow_m . For example, \Rightarrow_a , \Rightarrow_b , and \Rightarrow_* all show some welcome increase of possibility for accelerations above 1.0, but they still maintain a possibility of 0.67 for a less than 0.0 as in the graph for \Rightarrow_m , even though there is no possibility, physically speaking, that those accelerations would improve the standing of the pendulum. The three operators \Rightarrow_s , \Rightarrow_g , and \Rightarrow_Δ all show a welcome focus around the interval (1,2), which is physically meaningful, but they somewhat presumptuously assert the possibility of 1.0 (absolute truth) for values of a within the interval [1.5, 1.67], when neither of the contributing rules applied in this example with a certainty above 0.67, and neither of the conclusion fuzzy sets of those rules have degrees of membership over that interval above 0.67.



Of the fuzzy implication operators described, \Rightarrow_c performs the best, maintaining realistic possibilities for positive accelerations but not permitting negative accelerations. But for \Rightarrow_c to be the best is a paradox, because \Rightarrow_c is not considered to be a “real” fuzzy implication operator, since it does not correspond to the classical implication operator, unlike the others. *In truth, its perceived weakness turns out to be its strength.* \Rightarrow_c works better exactly because it corresponds to a conjunction operation, rather than an implication, and a conjunction better models the semantics of a crisp expert system rule. This shows the fundamental flaw in the classical formulation, that it attempts to define expert system reasoning based on logical implication.

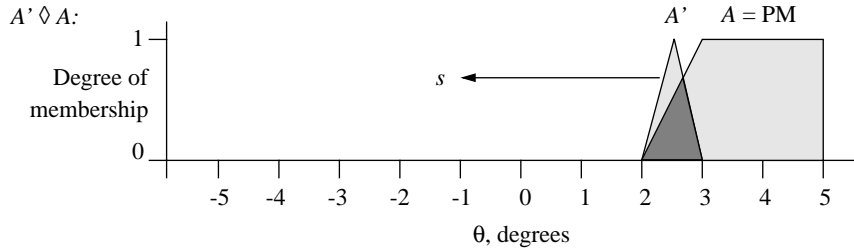
8.4. Satisfaction inferencing

By contrast, the new satisfaction approach to inferencing is considerably simpler. We will use the max-min satisfaction operator (\diamond_{\wedge}) for \diamond and scaled muting (∞_*) for ∞ . Taking the previous example of an inverted pendulum, A and B from rule 1, and A' “about” 2.5 as before, we compute

$$B' = (A' \diamond A) \infty B$$

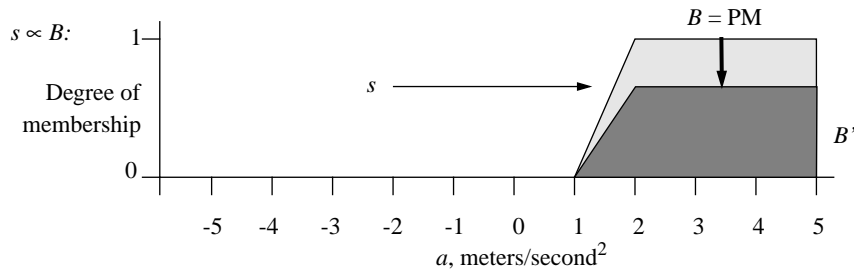
by first computing

$$\begin{aligned} s &= A' \diamond A \\ &= \bigcup_{\theta \in U} \min(A'(\theta), A(\theta)) \\ &= 0.67 \end{aligned}$$



Note that in comparison with the classical inferencing C , which was a fuzzy relation requiring a three-dimensional graph, this intermediate value s is simply a real number in $[0,1]$.

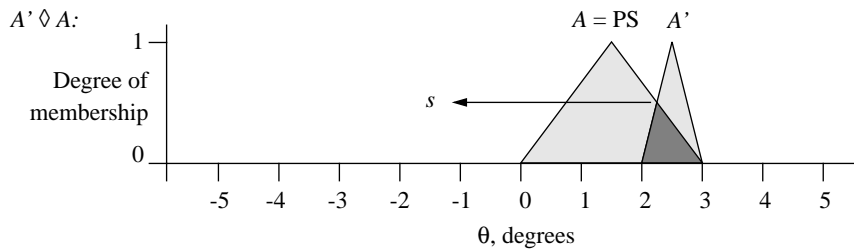
This value of s is then used to mute the fuzzy set B , generating the result $B' = s \circ B$. Using the scaled muting operator \circ_* , we create



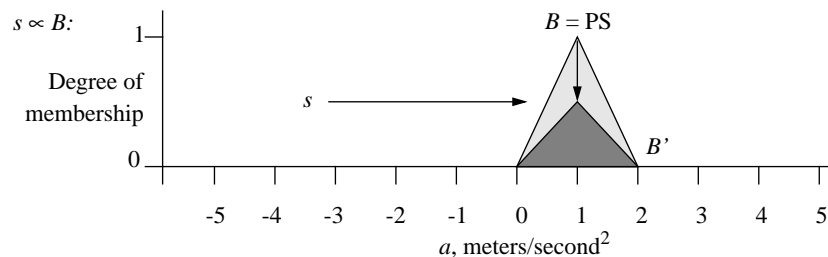
This B' has the intuitive advantages that it has a shape that is related to the original B ; that it neither exceeds the membership of B , nor the degree of satisfaction calculated; and it may be combined with the results of other fuzzy rules by using the f_{or} operator.

For example, the other rule that applies here (rule 2), where $A = \text{“}\theta \text{ is PS”}$ and $B = \text{“}a \text{ is PS”}$, yields the following satisfaction:

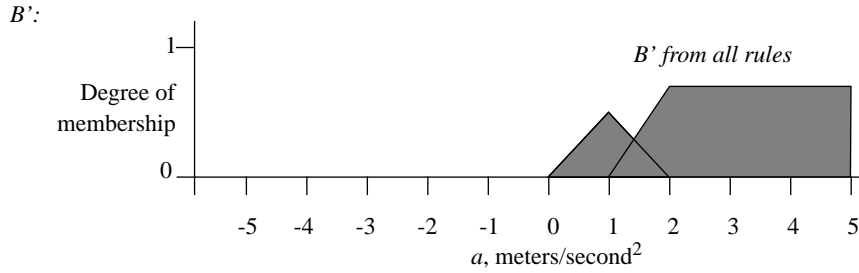
$$\begin{aligned}
 s &= A' \diamond A \\
 &= \bigcup_{\theta \in U} \min(A'(\theta), A(\theta)) \\
 &= 0.5
 \end{aligned}$$



which then produces the following contribution $B' = s \circ B$:

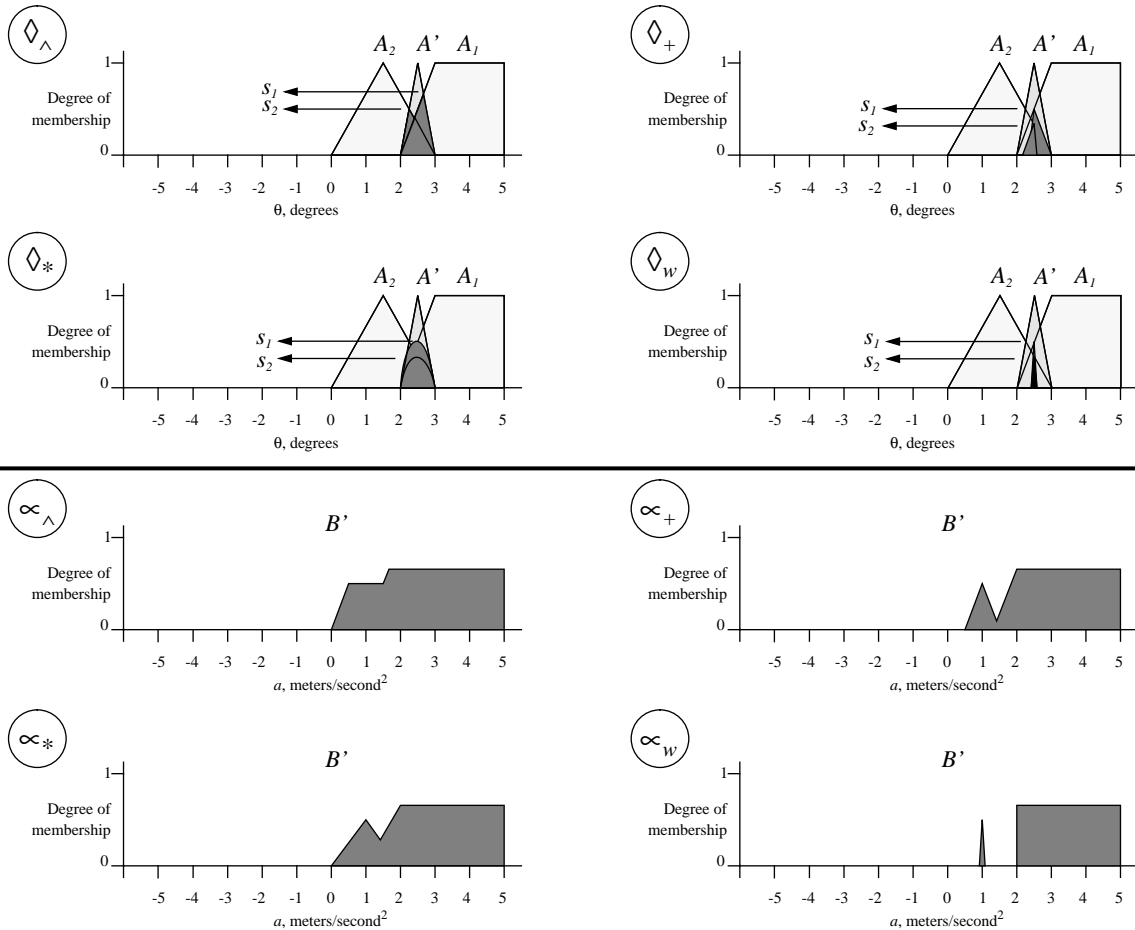


Combining the results of these two rules using f_{or} , we achieve the final resulting fuzzy set:



Comparing this result with the fuzzy set resulting from the classical approach, we see they differ primarily in that the satisfaction approach does not support any possibility to accelerations less than zero, whereas the classical approach yields a flat fuzzy set, where all possibilities are equal. This example with θ about 2.5 degrees describes a condition where the pendulum is tipped about 2.5 degrees forward. Intuitively, if the pendulum was tipped forward and falling forward, we would want the platform to also accelerate forward to push the pendulum back up. *The results of the satisfaction approach reflect this expectation, whereas the classical results do not.*

Just as we repeated the inference process for each of the classical inferencing operators, here are the results of using the different satisfaction-oriented inferencing operators for this example. The different choices for \diamond compute the satisfaction values as shown, and assuming the satisfaction values from the operator \diamond_{\wedge} , the various muting operators work as follows:



Note that the fuzzy set resulting from using ∞_{\wedge} is the same as that shown earlier for \Rightarrow_c . The leveled, scaled, and lowered muting operators all give intuitive results. The drastic muting operator, ∞_w , seems weaker, and the other operators may be preferred for actual inferencing. Nevertheless, even ∞_w seems to yield superior results to most of the classical results for this example.

9. SUMMARY AND CONCLUSIONS

The satisfaction-oriented approach to fuzzy reasoning arises from a very simple change at the heart of inferencing, rotating the operators. It contrasts with the model normally studied in the literature, being simpler and more intuitive. It also is the model being used by most practical implementations, though some of those implementors have felt the need to justify their systems by describing them in terms of the classical approach.

There is nothing *wrong* with the classical approach, as long as the designer of the rules understands the semantics being used. However, the classical approach is a fuzzy adaptation of the ideas of crisp logic and theorem proving, which have an inherently different semantics from the operational, dynamic semantics of expert systems. Expert system reasoning is not theorem proving in the classical sense. As a separate field, expert systems deserves a fuzzy theoretical foundation consistent with its core ideas.

We present this new approach to fuzzy reasoning to provide a solid, mathematically sound theoretical basis for the good work going on in current implementations. But beyond this, satisfaction-oriented fuzzy reasoning establishes a framework for a large new space of fuzzy reasoning, including new definitions of satisfaction and muting. This world invites investigation, displaying both a clean theory and real-world effectiveness. Both theoretical and practical exploration can now begin from this reoriented view.

APPENDIX A

THEOREM 1: The max-dot method cannot be expressed in the classical formulation.

To prove that the max-dot method cannot be expressed in the classical formulation, assume the contrary, that it can. Then there exist versions of \Rightarrow and \circ such that for *all* fuzzy sets A' , A , and B , with universes of discourse U , U , and V respectively,

$$A' \circ (A \Rightarrow B) = B' = \left(\bigcup (A' \cap A) \right) \cdot B \quad (1)$$

or, expanding (1) using the definitions of the operators,

$$\lambda v. \left(\bigcup_{u \in U} A'(u) * r(A(u), B(v)) \right) = \lambda v. \left(\left(\bigcup_{u \in U} \min(A'(u), A(u)) \right) \cdot B(v) \right) \quad (2)$$

for some operators $*$ and r . As this is true for all A' , A , and B , let us choose A' , A , and B that are constant functions, where $A'(u) = a'$, $A(u) = a$, and $B(v) = b$, where a' , a , $b \in [0,1]$, for all $u \in U$ and $v \in V$. Then the particular choices of u and v do not matter, and the maximum operators \bigcup have no variety of selection, so the statement (2) reduces to

$$a' * r(a, b) = \min(a', a) \cdot b \quad (3)$$

Since this is true for all a' , a , and b , consider the case where $a' = 1$:

$$1 * r(a, b) = \min(1, a) \cdot b$$

Since $\min(1, x) = x$ and $1 * x = x$ (due to the top identity property of $*$ as a triangular norm),

$$r(a, b) = a \cdot b$$

This shows that r must be exactly the multiplication operation. Thus we can rewrite (3) as

$$a' * (a \cdot b) = \min(a', a) \cdot b \quad (4)$$

Since this is true for all a' , a , and b , consider the case where $b = 1$:

$$\begin{aligned} a' * (a \cdot 1) &= \min(a', a) \cdot 1 \\ a' * a &= \min(a', a) \end{aligned}$$

This shows that $*$ must be exactly the minimum operator. So then (4) becomes

$$\min(a', a \cdot b) = \min(a', a) \cdot b \quad (5)$$

But this cannot be true! For example, choosing $a' = 0.4$, $a = 1$, $b = 0.5$ we have

$$\begin{aligned} \min(0.4, 1 \cdot 0.5) &= \min(0.4, 1) \cdot 0.5 \\ 0.4 &= 0.2 \end{aligned}$$

which is false. This false conclusion contradicts our assumption, so we have proven that there are no versions of \Rightarrow and \circ which can express Togai's max-dot inference method.
Q.E.D.

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